center. Since the exactness of asymptotic formulas (2.1) is of the order of $O(n^{\zeta_i})$, where $\zeta_i = (\gamma - 1) / (\gamma d)$ the accuracy of formulas (2.4) is of the order of $O(t^{-\delta\zeta_i})$. The Euler equation (2.7) is derived from (1.1) and (2.4) with an accuracy of the order of $O(t^{-\delta\zeta_E})$, where $\zeta_E = (2\gamma + d - 2) / (\gamma d)$. There are no further simplifications in the kinetic mode. Hence, Eqs. (2.10) and (2.11) define the K-mode in the neighborhood of the center with an accuracy of the order of $O(t^{-\lambda_K})$, where $\lambda_K = \min(\delta_K\zeta_i, \zeta_E\delta_K)$.

In the Planck-mode Eq. (2.10) is approximated by (3.7) with an accuracy of the order of $O(t^{\delta_\tau} \ln t^{-\delta_\tau})$. Hence solution (3.8) defines such flow with an accuracy of the order of

$$O\{\min[t^{-\delta_P\zeta_i}, t^{-\delta_P\zeta_E}, t^{\delta_\tau(\delta_P)} \ln t^{-\delta_\tau(\delta_P)}]\}$$

The reduction of (2.10) to (3.11) is achieved in the R-mode with an error of the order of $O(t^{-2\delta_\tau})$. The resulting error of determination of such flows is of the order of

$$O(t^{-\lambda_R}), \quad \lambda_R = \min[\delta_R\zeta_i, \delta_R\zeta_E, 2\delta_\tau(\delta_R)]$$

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METHOD OF SOLUTION OF CERTAIN BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC SYSTEMS OF QUASILINEAR EQUATIONS OF FIRST ORDER WITH TWO VARIABLES

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We propose a method of obtaining exact solutions of certain boundary value problems for hyperbolic systems of quasilinear equations of first order with two unknowns. The method utilizes special series. As an example, we solve the problem of motion of a plane, cylindrical or spherical piston in a gas with distributed density.

1. Let us consider the following system of equations:

$$A(x, \mathbf{U})\frac{\partial \mathbf{U}}{\partial t} + B(x, \mathbf{U})\frac{\partial \mathbf{U}}{\partial x} + C(x, \mathbf{U}) = 0 \tag{1.1}$$

$$\mathbf{U} = \{u_i(x, t)\}, \quad A(x, \mathbf{U}) = \{a_{ij}(x, \mathbf{U})\}, \quad B(x, \mathbf{U}) = \{b_{ij}(x, \mathbf{U})\}$$

$$\mathbf{C}(x, \mathbf{U}) = \{c_i(x, \mathbf{U})\}, \quad i, j = 1, \ldots, m$$

Let $\mathrm{U}^\circ(x) = \{u_1^\circ(x), \ldots, u_m^\circ(x)\}$ be the analytic solution of system (1.1). We assume that $a_{ij}(x, \mathrm{U})$, $b_{ij}(x, \mathrm{U})$ and $c_i(x, \mathrm{U})$ $(i, j = 1, \ldots, m)$ are functions analytic in the neighborhood of the point $M = \{x = 0, \ \mathrm{U} = \mathrm{U}^\circ\}$, and that the solution $\mathrm{U}^\circ(x)$ has a corresponding characteristic of system (1.1) which can always be written in the form

$$\varphi(x, t) = t + \Psi(x) = 0 \tag{1.2}$$

We also assume that $0 < |\varphi_x'| < \infty$, $\Psi(0) = 0$, and that the following $(m-1)$ th order determinant is not zero in the neighborhood of M:

$$\det\{a_{ij} + b_{ij}\varphi_x'\} \neq 0 \quad i, j = 2, \ldots, m$$

We seek a solution of system (1.1) with the following conditions:

$$\mathrm{U}(x, t)|_{\varphi(x, t)=0} = \mathrm{U}^\circ(x) \tag{1.3}$$

$$u_1(0, t) = \sum_{k=0}^{\infty} \xi_k t^k = H(t), \qquad u_1^\circ(0) = \xi_0$$

where ξ_k $(k = 0, \ldots, \infty)$ are specified constants.

The solution is sought in the neighborhood of the point $\{x = 0, \ t = 0, \ \varphi = 0\}$ for $t > 0$ and $\varphi > 0$.

Ludwig in [1] gave a method of solving the Cauchy problem for linear hyperbolic systems, using convergent expansions in terms of travelling waves. In these series generalized functions appear as multipliers containing all singularities of the order lower than that of the accompanied term. The coefficients of these generalized functions are obtained from ordinary differential equations. The proof of convergence of such series is reduced to the Cauchy-Kowalewska existence theorem.

Sidorov proposes in a number of his papers [2, 3] a method of obtaining exact solutions of certain boundary value problems for nonlinear hyperbolic equations of second order. The solutions are represented by special series in the hodograph space. The convergence of these series is proved by Bautin in [4] and the proof can also be reduced to the Cauchy-Kowalewska theorem. The present paper adjoins with the works mentioned above.

2. We seek the solution of the problem in the form of series

$$u_i(x, t) = \sum_{k=0}^{\infty} u_i^{(k)}(x)\,\varphi^k(x, t), \qquad i = 1, \ldots, m \tag{2.1}$$

Let us determine the coefficients of $u_i^{(k)}(x)$. First we expand the coefficients $a_{ij}(x, \mathrm{U})$, $b_{ij}(x, \mathrm{U})$ and $c_i(x, \mathrm{U})$ $(i, j = 1, \ldots, m)$ into series in the neighborhood of the point $\mathrm{U} = \mathrm{U}^\circ$. The expansion of $a_{ij}(x, \mathrm{U})$ has the form

$$a_{ij}(x, \mathrm{U}) = a_{ij}(x, \mathrm{U}^\circ) + \sum_{l_1 + \ldots + l_m = 1}^{\infty} \frac{\partial^{l_1 + \ldots + l_m} a_{ij}(x, \mathrm{U}^\circ)(u_1 - u_1^\circ)^{l_1} \ldots (u_m - u_m^\circ)^{l_m}}{\partial u_1^{l_1} \ldots \partial u_m^{l_m} l_1! \ldots l_m!}$$

We replace $u_i (i = 1, \ldots, m)$ in the above series by the series (2.1) and find from the resulting expression the coefficients $F(a_{ij}, k)$ accompanying φ^k $(k \geqslant 1)$. Let us denote by $f_{l_q}(k)$ the coefficient accompanying φ^k in the expansion in φ of the function of the form $(u_q - u_q^\circ)^{l_q}$ $(k \geqslant l_q \geqslant 1)$. This yields

$$f_{l_q}(k) = \sum_{\beta_1+\ldots+\beta_k=l_q} (u_q^{(1)})^{\beta_1}\ldots(u_q^{(k)})^{\beta_k}\, c^{l_q}_{\beta_1\ldots\beta_k}\delta^k_{\beta_1+2\beta_2+\ldots+k\beta_k}$$

$$c^{l_q}_{\beta_1\ldots\beta_k} = \frac{l_q!}{\beta_1!\ldots\beta_k!}$$

where δ_i^j is the Kronecker delta. Further, we introduce the notation

$$F_{l_i}(k) = \begin{cases} f_{l_i}(k), & k \geqslant l_i \geqslant 1 \\ 1, & l_i = 0 \\ 0, & k < l_i \end{cases}$$

We denote by $F_{l_1\ldots l_m}(k)$ the coefficient accompanying φ^k $(k \geqslant 1)$ in the expansion in φ of the function $(u_1 - u_1^0)^{l_1}\ldots(u_m - u_m^0)^{l_m}$. Then

$$F_{l_1\ldots l_m}(k) = \sum_{\Delta_1,\ldots,\Delta_{m-1}=0}^{k} F_{l_1}(\Delta_1)\ldots F_{l_{m-1}}(\Delta_{m-1})F_{l_m}(k-\Delta_1-\ldots-\Delta_{m-1})$$

In the above notation the coefficient $F(a_{ij}, k)$ has the form

$$F(a_{ij}, k) = \sum_{l_1+\ldots+l_m=1}^{k} \frac{\partial^{l_1+\ldots+l_m} a_{ij}(x, \mathrm{U}^\circ)\, F_{l_1\ldots l_m}(k)}{\partial u_1^{l_1}\ldots\partial u_m^{l_m}\, l_1!\ldots l_m!}$$

The coefficients $b_{ij}(x, \mathrm{U})$, and $c_i(x, \mathrm{U})$ can be obtained in the same manner.

Thus, replacing u_j $(j = 1, \ldots, m)$ in the i-th equation of the system (1.1) by the series (2.1), we obtain an expression with a series in its left-hand side, and the coefficient accompanying φ^k in this series can be found from the formula

$$R_{ki} = \sum_{j=1}^{m}\sum_{\Delta=0}^{k}\Big[(\Delta+1)u_j^{(\Delta+1)}F(a_{ij}, k-\Delta) +$$
$$(\Delta+1)u_j^{(\Delta+1)}F(b_{ij}, k-\Delta)\varphi_x' + \frac{\partial u_j^{(\Delta)}}{\partial x}F(b_{ij}, k-\Delta)\Big] + F(c_i, k)$$
$$k = 0, \ldots, \infty$$

The sufficient condition for the series (2.1) to be a formal solution of the system (1.1) is, that

$$R_{ki} = 0, \quad i = 1, \ldots, m; \; k = 0, \ldots, \infty \tag{2.2}$$

The system (2.2) can be written in the form

$$(k+1)\, A_x(\mathrm{U}^{(k+1)}) + L_k(\mathrm{U}^{(k)}, x) = 0, \; k = 0, \ldots, \infty$$

where

$$L_k = \{(L_k)_i\}, \qquad i = 1, \ldots, m$$

$$(L_k)_i = \sum_{j=1}^{m}\Big[\sum_{\Delta=0}^{k-1} F(a_{ij}, k-\Delta)\,u_j^{(\Delta+1)}(\Delta+1) +$$
$$F(b_{ij}, k-\Delta)(\Delta+1)u_j^{(\Delta+1)}\varphi_x' + \sum_{\Delta=0}^{k} F(b_{ij}, k-\Delta)\frac{\partial u_j^{(\Delta)}}{\partial x}\Big] + F(c_i\, k)$$

$$A_x = \{a_{ij}(x, \mathrm{U}^\circ) + b_{ij}(x, \mathrm{U}^\circ)\varphi_x'\}, \quad i,j = 1, \ldots, m$$

(A_x is a degenerate matrix, since $\varphi = 0$ is the characteristic).

Using the fact that $U^\circ(x)$ is a solution of the system (1.1), we can obtain the following system of equations for determining $u_i^{(k)}(x)$:

$$\begin{aligned} &A_x \mathbf{U}^{(1)} = 0 \\ &L_1(\mathbf{U}^{(1)}, x) + 2A_x\mathbf{U}^{(2)} = 0 \\ &\ldots\ldots\ldots\ldots\ldots \\ &L_k(\mathbf{U}^{(k)}, x) + (k+1)A_x\mathbf{U}^{(k+1)} = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2.3}$$

The system (2.3) differs from the equations obtained by Ludwig in [1] in the form of the operator L_k. Following Ludwig we solve the system (2.3) by finding the fight ($\mathbf{r}(x)$) and the left ($\mathbf{d}(x)$) null-vectors of the matrix A_x. Since $\det A_x = 0$, these vectors are nonzero. Let us assume that

$$\sum_{i,j} d_i(x) b_{ij}(x, \mathbf{U}^\circ) r_j(x) > 0$$

near the point M. Next we set $\mathbf{U}^{(1)} = \sigma_1(x)\,\mathbf{r}(x)$, where $\sigma_1(x)$ is a scalar multiplier. We determine $\sigma_1(x)$ so that the system described by the second line in (2.3) is compatible. A solution of this system exists, if $\mathbf{d}(x) L_1(\sigma_1\mathbf{r}, x) = 0$. Let us write this equation in a different form

$$\partial\sigma_1/\partial x + G(x)\,\sigma_1 = Q(x)\,\sigma_1^2 \tag{2.4}$$

$$G(x) = \frac{1}{\Sigma_0}(\Sigma_1 + \Sigma_2 + \Sigma_3), \qquad Q(x) = \frac{1}{\Sigma_0}\Sigma_4$$

$$\Sigma_0 = \sum_{i,j=1}^m d_i b_{ij}(x, \mathbf{U}^\circ) r_j, \quad \Sigma_1 = \sum_{i,j=1}^m d_i b_{ij}(x, \mathbf{U}^\circ)\frac{\partial r_j}{\partial x}.$$

$$\Sigma_2 = \sum_{i,j,q=1}^m d_i \frac{\partial b_{ij}(x, \mathbf{U}^\circ)}{\partial u_q} r_q \frac{\partial \overset{\circ}{u}_j}{\partial x}, \quad \Sigma_3 = \sum_{i,q=1}^m d_i \frac{\partial c_i(x, \mathbf{U}^\circ)}{\partial u_q} r_q$$

$$\Sigma_4 = -\sum_{i,j,q=1}^m d_i \left[\frac{\partial a_{ij}(x, \mathbf{U}^\circ)}{\partial u_q} + \frac{\partial b_{ij}(x, \mathbf{U}^\circ)}{\partial u_{qi}}\right] r_q r_j$$

Integrating (2.4) we obtain

$$\sigma_1(x) = \exp\left(-\int G(x)\,dx\right)\left[\gamma_1 - \int \exp\left(-\int G(x)\,dx\right) Q(x)\,dx\right]^{-1}$$

Function $\mathbf{U}^{(2)}$ is sought in the form

$$\mathbf{U}^{(2)} = \sigma_2(x)\,\mathbf{r}(x) + \mathbf{h}_2(x) \qquad \mathbf{h}_2(x) = \{h_2^i(x)\}$$

where $\mathbf{h}_2(x)$ is determined from the system

$$L_1(\mathbf{U}^{(1)}, x) + 2A_x\mathbf{h}_2 = 0$$

which is compatible.

Function $\sigma_2(x)$ is found from the relation $\mathbf{d}(x) L_2(\sigma_2\mathbf{r} + \mathbf{h}_2, x) = 0$, which can be written as

$$\partial\sigma_2/\partial x + G_1(x)\,\sigma_2 = Q_1(x)$$

The general solution of this equation can be easily obtained.

The functions $\sigma_k(x)$ $(k > 2)$ are determined successively in the manner similar to that used to obtain $\sigma_2(x)$. The arbitrary constants γ_k are determined from the condition

$$\sum_{k=0}^{\infty} \xi_k t^k = \sum_{k=0}^{\infty} [\sigma_k r_1 t^k + h_k^1 t^k]$$

where $r_1 \neq 0$, since $\det\{a_{ij} + b_{ij}\varphi_x'\} \neq 0$, $i, j = 2, \ldots, m$. The convergence of the series obtained follows from [4].

The above procedure holds also for the case when the coefficients of the initial system (1.1) depend on t, and for the case of equations with many variables. In these cases σ_k are obtained from the partial differential equations.

3. Let a gas at rest $(u = \mu_0(x) \equiv 0)$ of density $\rho = \alpha_0(x)$ and entropy $S = S_0(x)$ be situated either outside a spherical (or cylindrical) piston of radius x_0, or to the right of a plane piston situated at the distance x_0 from the coordinate origin. The quantities $\alpha_0(x)$, $\mu_0(x)$ and $S_0(x)$ satisfy the following system of equations:

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} = -\frac{\nu\rho u}{x} \tag{3.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial S}\frac{\partial S}{\partial x} = 0$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0$$

where ν and $P = P(\rho, S)$ are taken in accordance with [5].

At the instant $t = 0$ the piston begins to move into the gas (the piston radius increases) at zero initial velocity and nonzero acceleration. The motion of the gas is described by the system (3.1). We seek a solution of this system near the point $\{\varphi = 0,\ t = 0,\ x = x_0\}$, where $\varphi = 0$ is the characteristic corresponding to the solution $\alpha_0(x)$, $\mu_0(x)$, $S_0(x)$. The gas flow satisfies the following boundary conditions:

$$u(x, t)|_{\varphi=0} = 0, \quad \rho(x, t)|_{\varphi=0} = \alpha_0(x) \tag{3.2}$$

$$S(x, t)|_{\varphi=0} = S_0(x), \quad u(x(t), t) = \partial x(t)/\partial t$$

$$(x(t) = x_0 + \xi_1 t^2 + \ldots,\ \xi_1 > 0)$$

here $(x(t)$ is the law of motion of the piston. The solution of the problem (3.1), (3.2) is sought in the form

$$\rho(x, t) = \sum_{k=0}^{\infty} \alpha_k(x)\varphi^k(x, t), \qquad u(x, t) = \sum_{k=1}^{\infty} \mu_k(x)\varphi^k(x, t)$$

$$S(x, t) = \sum_{k=0}^{\infty} S_k(x)\varphi^k(x, t)$$

Let us carry out the computations for the case

$$P(\rho, S) = S\rho^2, \quad S_0 = \alpha_0^{-2}, \quad x(t) = x_0 + \xi_1 t^2$$

In this case the system (3.1) becomes

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=-\frac{\nu\rho u}{x} \qquad (3.3)$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+2S\frac{\partial\rho}{\partial x}+\rho\frac{\partial S}{\partial x}=0$$
$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}=0$$

The matrix A_x for (3.3) has the form

$$A_x=\left\|\begin{matrix}-1 & \alpha_0\varphi_x' & 0\\ 2\alpha_0^{-2}\varphi_x' & -1 & \alpha_0\varphi_x'\\ 0 & 0 & -1\end{matrix}\right\|$$

The right and left null-vectors are, respectively,

$$\mathbf{r}(x)=\left\|\begin{matrix}2^{-1/2}\alpha_0^{3/2}(x)\\ 1\\ 0\end{matrix}\right\|,\quad \mathbf{d}(x)=\left\|\begin{matrix}2^{1/2}\alpha_0^{-3/2}(x)\\ 1\\ 2^{-1/2}\alpha_0^{1/2}(x)\end{matrix}\right\|$$

The following relations yield $\alpha_1(x)$, $\mu_1(x)$ and $S_1(x)$:

$$-2\alpha_2+2\mu_1\alpha_1\varphi_x'+2\alpha_0\mu_2\varphi_x'+\frac{\nu\alpha_0\mu_1}{x}+\frac{\partial\alpha_0}{\partial x}\mu_1+\frac{\partial\mu_1}{\partial x}\alpha_0=0$$
$$-2\mu_2+\mu_1^2\varphi_x'+2S_0\frac{\partial\alpha_1}{\partial x}+4S_0\alpha_2\varphi_x'+\frac{\partial S_0}{\partial x}\alpha_1+2S_2\alpha_0\varphi_x'=0$$
$$-2S_2+\mu_1S_1\varphi_x'+\frac{\partial S_0}{\partial x}\mu_1=0$$

In accordance with Sect. 2 we assume that

$$\alpha_1(x)=\sigma_1(x)\,2^{-1/2}\alpha_0^{3/2}(x),\quad \mu_1(x)=\sigma_1(x),\quad S_1(x)=0$$

We find $\sigma_1(x)$ from the equation

$$2^{3/2}\alpha_0^{-1/2}\frac{\partial\sigma_1}{\partial x}+3\alpha_0^{1/2}2^{-1/2}\sigma_1^2+\left(2^{1/2}\frac{\nu}{x}\alpha_0^{-1/2}+2^{-1/2}\alpha_0^{-3/2}\frac{\partial\alpha_0}{\partial x}\right)\sigma_1=0$$

which on integration gives

$$\sigma_1(x)=\alpha_0^{-1/4}(x)\,x^{-\nu/2}\left[\gamma_1+{}^3/_4\int\alpha_0^{3/4}(x)\,x^{-\nu/2}\,dx\right]^{-1}$$

Let $\alpha_0(x)=\alpha_0=$ const and $x(t)=x_0+\xi_1t^2$. Then

$$\sigma_1(x)=\left[x^{\nu/2}\left(\gamma_1+{}^3/_4\alpha_0\frac{x^{1-\nu/2}}{1-\nu/2}\right)\right]^{-1}\quad(\nu=0,1)$$
$$\sigma_1(x)=[x(\gamma_1+{}^3/_4\alpha_0\ln x)]^{-1}\quad(\nu=2)$$
$$\gamma_1=-{}^1/_4\left(2x_0^{-\nu/2}\xi_1^{-1}+3\alpha_0\frac{x_0^{1-\nu/2}}{1-\nu/2}\right)\quad(\nu=0,1)$$
$$\gamma_1=-{}^1/_4(2x_0^{-1}\xi_1^{-1}+3\alpha_0\ln x_0)\quad(\nu=2)$$

The position x^* at which the infinite gradients appear is determined by equating to zero the expressions for $\sigma_1(x)$ contained within the round brackets, and the time $t^*(t^*>0)$ of their appearance is found from the equation

$$\varphi(x^*,t)=0$$

In the case under consideration (a piston moving into the gas, $\xi_1 > 0$) the infinite gradients always appear. If the initial speed of sound in the gas is equal to unity, we obtain

$$t^* = {}^2/_3 w, \quad w = 2\xi_1 \quad (\nu = 0)$$

$$t^* = \left[6 + \frac{1}{wx_0}\right] \frac{1}{9w} \quad (\nu = 1)$$

For this case analogous formulas are obtained in [3].

Let now $\rho = \alpha_0(x)$ and $S = S_0(x)$. Then we have

$$\gamma_1 = -{}^1/_4 \left(2\xi_1^{-1} x_0^{-\nu/2} \alpha_0^{-1/4}(x_0) + 3 \int \alpha_0^{3/4}(x)\, x^{-\nu/2}\, dx \,|_{x=x_0}\right)$$

The sufficient condition for the appearance of infinite gradients is

$$\int \alpha_0^{3/4}(x)\, x^{-\nu/2}\, dx \to \infty, \quad x \to \infty$$

Let $\nu = 1$, $\alpha_0(x) = x^2$ and $x(t) = 1 + {}^{20}/_3 t^2$. We obtain

$$\sigma_1(x) = [0.75\ x\,(0.6 - 0.5\ x^2)]^{-1}, \quad x^* = 1.1, \quad t^* = 0.074$$

$$\rho(x, t) = x^2 \left[1 + 8^{1/2} / 3(0.5\ x^2 - 0.6)^{-1}(-t - 8^{-1/2} + x^2 8^{-1/2})\right] + \ldots$$

The expressions for $\sigma_2(x)$ and $h_2(x)$ are bulky and therefore are omitted, $u(x, t)$ and $S(x, t)$ have the same form as before.

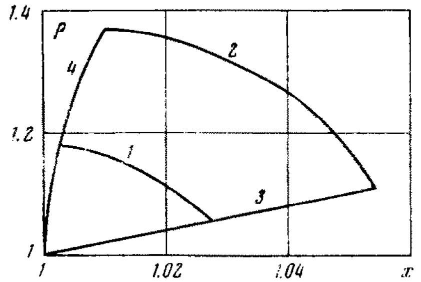

Fig. 1

Figure 1 depicts the results of numerical computations for the density $\rho(x, t)$ using three terms of the expansion for $t = 0.02$ and $t = 0.04$, corresponding to the curves 1 and 2. The curve 3 gives the initial density distribution and curve 4 defines the position of the piston. The maximum values of ρ and the corresponding values of x are as follows: (1.18, 1.0025) and (1.372, 1.011).

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